

NOTE ON A ONE-DIMENSIONAL SYSTEM OF ANNIHILATING PARTICLES

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Abstract. We consider a system of annihilating particles where particles start from the points of a Poisson process on either the full-line or positive half-line and move at constant i.i.d. speeds until collision. When two particles collide, they annihilate. We assume the law of speeds to be symmetric. We prove almost sure annihilation of positive-speed particles started from the positive half-line, and existence of a regime of survival of zero-speed particles on the full-line in the case when speeds can only take 3 values. We also state open questions.

1. INTRODUCTION

Let us first define informally the model that we are working on. Particles are released from the locations of a Poisson point process on either the full-line \mathbb{R} or the half-line \mathbb{R}_+ with i.i.d. velocities sampled from a distribution μ with bounded support. Each particle moves at constant velocity, and when two particles collide, they annihilate. We are interested in the possible survival of some particles forever.

In this note, we treat the case of a symmetric distribution μ (i.e. $v \stackrel{(d)}{=} -v$ if v has law μ) for particles starting from the half-line, and the case of a symmetric distribution μ on $\{-1, 0, 1\}$ for particles starting from the full-line. For symmetric distributions μ , and particles starting on the full-line, only 0-speed particles may survive. The question on the half-line is more intriguing. In Section 3, we present an argument based on symmetry to conclude that positive-velocity particles annihilate almost surely. We believe, but could not prove, that negative-velocity particles have positive chance to survive. On the full-line, we are specifically interested in the symmetric discrete velocity case $\mu = \frac{1-p}{2}\delta_{-1} + p\delta_0 + \frac{1-p}{2}\delta_1$, where p is a parameter in $[0, 1]$. In this case, numerical evidence suggests that there is a phase transition: for p small (approximately for $p < 0.25$), every particle annihilates almost surely but particles keep crossing 0 at arbitrarily large times, while for larger values of p , 0-speed particles manage to survive with positive probability. In Section 4, we present a simple proof of survival at $p > 1/3$ and discuss some extensions. We could not prove that particles die at small p , and leave it as another open question.

This model relates to the so-called “bullet problem”, which emerged as a challenge problem by David Wilson on IBM website [1]. In the “bullet” model, particles are released from 0 at times of a Poisson point process, with i.i.d. positive velocities $(w_k)_{k \geq 1}$, and annihilate on collision. It was conjectured that, if the law of speeds is uniform on $[0, 1]$, there is a velocity $v_c > 0$ such that bullets slower than v_c annihilate almost surely, while faster ones may survive. Compared to our model on \mathbb{R}_+ , the “bullet” model corresponds to switching time with space, which results in replacing speeds w_k by $v_k = 1/w_k$. Our results on the half-line can therefore be rephrased for this other model. The present version of the model however enjoys invariance by linear transforms of speeds and symmetry properties that make it more suitable for study.

We learned before publishing that other authors [2] independently obtained results similar to ours.

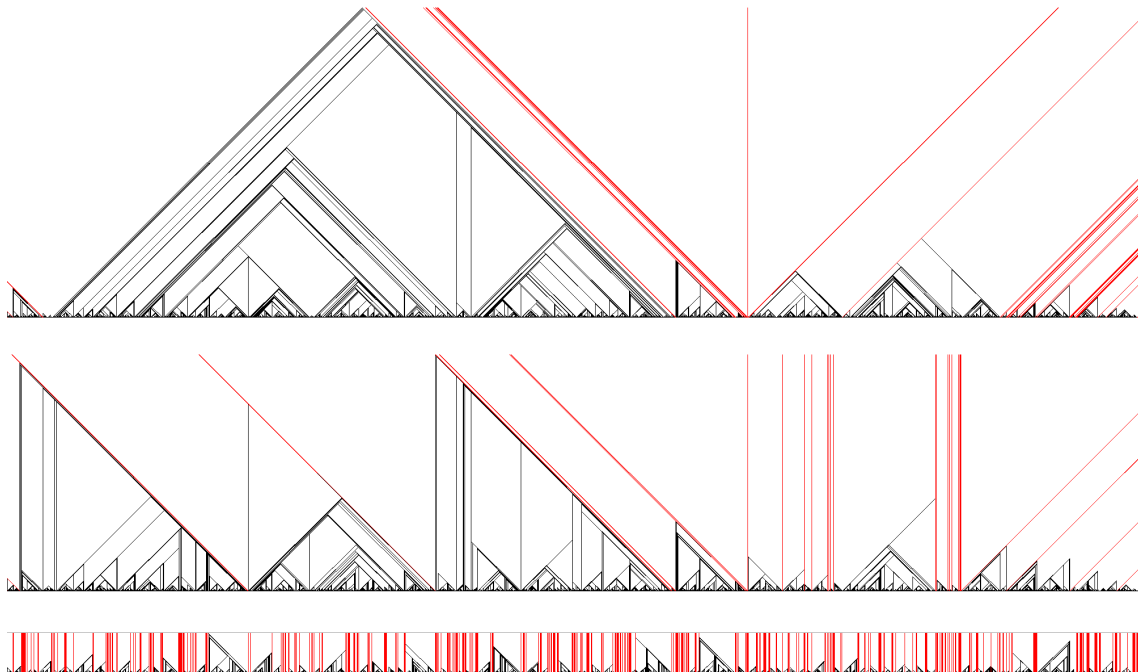


FIGURE 1. Simulations for $\mu = \frac{1-p}{2}\delta_{-1} + p\delta_0 + \frac{1-p}{2}\delta_1$, with $p = 0.24$ (top), $p = 0.25$ (middle) and $p = 0.26$ (bottom), with the horizontal direction as space and vertical (up) as time. The bottom picture contains 10 million particles. Note that, as can be guessed, the pictures are coupled by removing some 0-speed particles from the lower configuration, hence pictures have slightly different horizontal scale (or Poisson intensity).

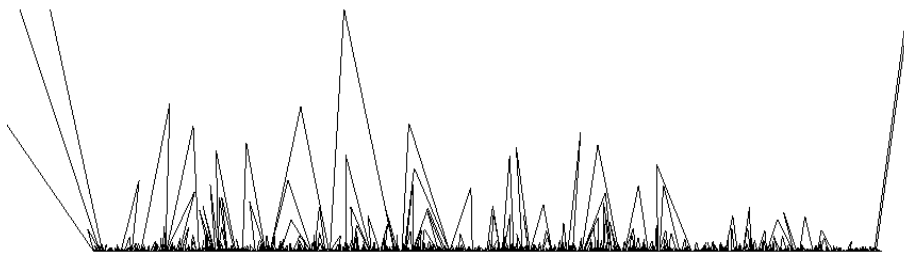


FIGURE 2. Simulation for μ uniform on $[-1, 1]$ and 10^5 particles.

2. DEFINITIONS AND BASIC PROPERTIES

Let $(x_k)_{k \in \mathbb{Z}}$ be a Poisson point process with intensity 1 on \mathbb{R} under Palm measure, i.e. conditioned on containing 0:

$$\dots < x_{-1} < x_0 = 0 < x_1 < \dots,$$

which is equivalent to $(x_k)_{k \geq 1}$ and $(-x_{-k})_{k \geq 1}$ being independent Poisson point processes on \mathbb{R}_+ . The points x_k are meant as starting locations of particles.

Let μ be a distribution on \mathbb{R} with bounded support. Let $(v_k)_{k \in \mathbb{Z}}$ be i.i.d. random variables with law μ , independent of $(x_k)_{k \in \mathbb{Z}}$, standing for the velocities of the particles.

Let us denote by \mathbb{P}_μ the law of $(x_k)_k, (v_k)_k$. We shall write \mathbb{P}_p in the particular case when $\mu = \frac{1-p}{2}\delta_{-1} + p\delta_0 + \frac{1-p}{2}\delta_1$, for $p \in [0, 1]$.

The process is defined as follows. From each location x_k , a particle is released at time 0, with speed v_k (hence going to the right if $v_k > 0$, to the left otherwise). Particles move at constant speed until they collide with another particle, at which point both annihilate and therefore don't take part in later collisions. Due to interdistances between starting locations having an atomless distribution and being independent of speeds, almost surely no triple collision happens. The model would be obviously well-defined if only finitely many particles are considered, for there is a chronologically first collision to be dealt with. In the case when particles initially lie on \mathbb{R}_+ , well-definedness is still ensured if speeds are lower bounded: for $i, j \in \mathbb{Z}_+$, given x_i, x_j, v_i, v_j , the collision of i and j can indeed be checked by only considering the system formed by the finitely many particles on the left of the rightmost location where a triple collision with i and j could be triggered. Using symmetry, collisions are then well-defined on \mathbb{R} since speeds are assumed bounded.

For $k, l \in \mathbb{Z}$ belonging to an interval $I \subset \mathbb{R}$, we denote by $k \xleftrightarrow{I} l$ the property that the particles released from x_k and x_l mutually annihilate in the system restricted to particles departing from $\{x_i : i \in I\}$. We denote by $k \xleftrightarrow{I} \infty$ the property that the particle from x_k survives, i.e. is not annihilated by another particle, in this restricted system.

Note that, for given starting locations, collision events $\{i \xleftrightarrow{I} j\}$ only depend on ratios of differences of speeds, so that adding the same amount to every speed (i.e. shifting μ), or multiplying them by a constant, does not change the probability of any event relating to collisions. We shall refer to this simple fact as the *linear speed-change invariance property*. In particular, several “bullet” models (see Introduction) are in correspondence with a given μ , through $w_k \mapsto v_k = 1/(\alpha + \beta w_k)$ for any α, β such that $\alpha + \beta w_k > 0$ a.s..

Also, a symmetry property obviously holds with respect to the reflection $x \mapsto -x$, namely that the law of the model with particles starting from $I \subset \mathbb{R}$ is the same as that of the reflection of the model with particles starting from $-I$ with speeds sampled from $\mu(-dv)$.

3. EXTINCTION FOR SYMMETRIC DISTRIBUTIONS

Proposition 1. *Assume μ is symmetric, $\mu \neq \delta_0$.*

- a) *For any $v > 0$, $\mathbb{P}_\mu(0 \xleftrightarrow{\mathbb{R}_+} \infty \mid v_0 = v) = 0$, where the conditioning here is understood as setting $v_0 = v$ and letting $(v_k)_{k \neq 0}$ be i.i.d. with law μ .*
- b) *If $\mathbb{P}_\mu(0 \xleftrightarrow{\mathbb{R}} \infty) = 0$, then almost surely $N_+ = \infty$, where N_+ is the number of particles that ever cross 0 in the system restricted to \mathbb{R}_+ .*

This Proposition will follow from the two lemmas below.

Lemma 1. *For $x \in \mathbb{R}$, let $N_x = \#\{i, j \in \mathbb{Z} \mid i < x < j, i \xleftrightarrow{\mathbb{R}} j\}$ denote the number of couples of particles that meet over x . If $\mathbb{P}_\mu(0 \xleftrightarrow{\mathbb{R}} \infty) = 0$, then a.s., for all x , $N_x = \infty$.*

Proof of Lemma 1. We use a parity argument inspired by [3, page 43]. First note that the event $\{N_x = \infty\}$ does not depend on x , is translation invariant, and therefore has probability 0 or 1 due to ergodicity. Assume that $\mathbb{P}_\mu(0 \xleftrightarrow{\mathbb{R}} \infty) = 0$ and, by contradiction,

that $N_{x_1/2} < \infty$ almost surely. Let us view the trajectories of particles as space-time curves (actually, line segments) in the upper half-plane \mathbb{H} (cf. Figures 1 and 2). Since $\mathbb{P}_\mu(0 \xleftrightarrow[\mathbb{R}]{} \infty) = 0$, almost surely for every $i \in \mathbb{Z}$, there is $j \in \mathbb{Z}$ such that $i \xleftrightarrow[\mathbb{R}]{} j$, and the trajectories of i and j split \mathbb{H} into one finite and one infinite components. Note also that all these pairs of trajectories are disjoint of each other. Since furthermore $N_{x_1/2} < \infty$ a.s., the union of these curves delimitates exactly one infinite component in \mathbb{H} . This implies that the event $\{N_{x_1/2} \text{ is even}\}$ is invariant by even shifts, i.e. by replacing $(x_k, v_k)_{k \in \mathbb{Z}}$ by $(x_{k+2} - x_2, v_{k+2})_{k \in \mathbb{Z}}$. $N_{x_1/2}$ is indeed the number of curves that separate $\frac{x_1}{2}$ from the infinite component, and a shift amounts to crossing a boundary and thus increases or decreases by 1 the number of curves separating from the infinite component, hence changing parity. By ergodicity, this event therefore has to have probability 0 or 1. However, its probability is $1/2$ due to shift invariance and alternance of parities. The conclusion follows from this contradiction. \square

Lemma 2. *Assume μ is symmetric, $\mu \neq \delta_0$. Then $\mathbb{P}_\mu(0 \xleftrightarrow[\mathbb{R}]{} \infty \mid v_0 \neq 0) = 0$.*

Proof of Lemma 2. If $\mathbb{P}_\mu(v_0 > 0, 0 \xleftrightarrow[\mathbb{R}]{} \infty) > 0$ then, by ergodicity, almost surely a positive density of particles survive and have a positive speed, and by symmetry the same holds for negative speeds, contradicting their mutual survival. \square

Proof of Proposition 2. Let us first prove a). If $\mathbb{P}_\mu(0 \xleftrightarrow[\mathbb{R}]{} \infty) > 0$, then by Lemma 2 surviving particles have 0 speed, and by ergodicity a positive density of them survive, which prevents any positive speed particle from surviving. We can thus now suppose that $\mathbb{P}_\mu(0 \xleftrightarrow[\mathbb{R}]{} \infty) = 0$. Assume by contradiction that there is $v > 0$ such that $\mathbb{P}_\mu(0 \xleftrightarrow[\mathbb{R}_+]{} \infty \mid v_0 = v) > 0$, which we can, without loss of generality, take small enough so that $\mu((v, +\infty)) > 0$. We have by symmetry that $\mathbb{P}_\mu(0 \xleftrightarrow[\mathbb{R}_-]{} -\infty \mid v_0 = -v) > 0$ and thus, by independence, with positive probability we may have (cf. Figure 3), at the same time, $v_0 > v$, $v_1 < 0$, a $(-v)$ -speed particle at 0 would survive \mathbb{R}_- and a v -speed particle at x_1 would survive \mathbb{R}_+ . On this event, both a (virtual) v -speed particle and a (virtual) $(-v)$ -speed particle launched at 0 would survive, counting with the particle already at 0. Since this has positive probability, by ergodicity we conclude that almost surely there is a positive density of Poisson points where this happens. This implies that $N_0 < \infty$, for if j, k are indices of such points, with $j < 0 < k$, then the surviving trajectories of a $(-v)$ particle at k and of a v -particle at j bound the couples straddling 0 to lie between j and k and thus be finitely few (cf. Figure 3). Lemma 1 yields a contradiction.

Let us now consider b). Suppose $\mathbb{P}_\mu(0 \xleftrightarrow[\mathbb{R}]{} \infty) = 0$. Assume by contradiction that $\mathbb{P}_\mu(N_+ < \infty) > 0$. One may notice that $N_+ = N_- < \infty$ doesn't automatically imply that $N < \infty$. However, there is $k \in \mathbb{N}$ and rational locations $u_1 < v_1 < u_2 < v_2 < \dots < u_k < v_k$ such that, with positive probability, $N_+ = k$ and these k particles cross 0 at times $\tau_1 \in [u_1, v_1], \dots, \tau_k \in [u_k, v_k]$. By symmetry, the same event relative to \mathbb{R}_- has the same positive probability. We can then produce a positive probability event such that (cf. Figure 3) the above happens for the process on the right of x_k and on the left of x_{-k+1} respectively, and such that the particles from 1 to k (resp. from 0 to $-k+1$) meet the k particles that arrive from the right (resp. from the left). However we then have $N_0 = 0$, contradicting Lemma 1. \square

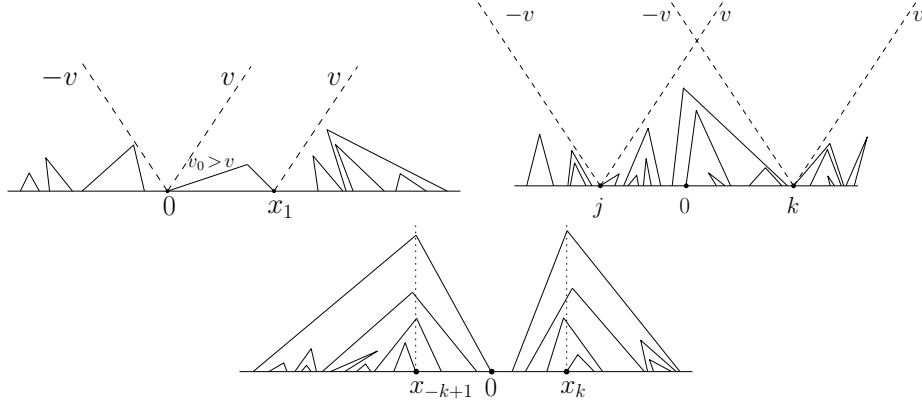


FIGURE 3. Illustrations of the proof of Proposition 2 a) (top) and b) (bottom).

Let us note that the following weaker result, namely the extinction of a positive-speed particle at 0 when the law of its speed is μ , is easier:

Proposition 2. *Assume μ is symmetric $\mu \neq \delta_0$. Then $\mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 > 0) = 0$.*

Proof of the proposition. By contradiction, assume $\mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 > 0) > 0$. Then we may find $v > 0$ such that

$$(1) \quad \mathbb{P}_\mu(v_0 \in (0, v]) > 0 \quad \text{and} \quad \mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 = v) > 0,$$

where the conditioning here is merely understood as letting $v_0 = v$ and letting $(v_k)_{k \neq 0}$ be i.i.d. with law μ . Indeed, one may for instance take v to be the median or any quantile of $\mathbb{P}_\mu(v_0 \in \cdot \mid 0 \longleftrightarrow \infty, v_0 > 0)$, for then $\mathbb{P}_\mu(0 < v_0 \leq v) > 0$ and $\mathbb{P}_\mu(v_0 \geq v, 0 \longleftrightarrow \infty) > 0$, and the last probability is smaller than $\mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 = v)$ because replacing the speed v_0 at x_0 by $v \leq v_0$ preserves survival on \mathbb{R}_+ .

Then (1), with its symmetric $\mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 = -v) > 0$, yields, for any $w \in [-v, v]$,

$$\begin{aligned} \mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 = w) &= \mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 = w) \mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 = w) \\ &\geq \mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 = v) \mathbb{P}_\mu(0 \longleftrightarrow \infty \mid v_0 = -v) > 0. \end{aligned}$$

Hence in particular, since $\mu((0, v]) > 0$ by (1),

$$\mathbb{P}_\mu(0 \longleftrightarrow \infty, v_0 \in (0, v]) > 0,$$

which contradicts Lemma 2. This proves the proposition. \square

We may remark that Lemma 2 ensures that the assumption of Proposition 2 b) is satisfied in particular if $\mu(\{0\}) = 0$. Also, the arguments of the proof of this lemma show that surviving particles on \mathbb{R} have the same speed, which is deterministic and therefore has to be an atom of μ :

Lemma 3. *If $\mathbb{P}_\mu(0 \longleftrightarrow \infty) > 0$, then there is $v \in \mathbb{R}$ such that $\mathbb{P}_\mu(v_0 = v \mid 0 \longleftrightarrow \infty) = 1$.*

Proof. Assume $\mathbb{P}_\mu(0 \xleftrightarrow[\mathbb{R}]{} \infty) > 0$. Let ν denote the law of v_0 given $\{0 \xleftrightarrow[\mathbb{R}]{} \infty\}$. Let $w \in \mathbb{R}$. If $\mathbb{P}_\mu(v_0 > w, 0 \xleftrightarrow[\mathbb{R}]{} \infty) > 0$ and $\mathbb{P}_\mu(v_0 < w, 0 \xleftrightarrow[\mathbb{R}]{} \infty) > 0$, then, by ergodicity, almost surely a positive density of particles survive and have a speed $> w$, and similarly with speed $< w$, which is contradicting their mutual survival. Therefore, either $\nu((w, +\infty)) = 0$ or $\nu((-\infty, w)) = 0$. Hence ν has to be a Dirac measure, for otherwise taking w to be its median yields a contradiction. \square

4. SURVIVAL FOR 3-SPEEDS DISTRIBUTIONS

Recall that \mathbb{P}_p refers to \mathbb{P}_μ where $\mu = \frac{1-p}{2}\delta_{-1} + p\delta_0 + \frac{1-p}{2}\delta_1$.

Proposition 3. *Assume $p > \frac{1}{3}$. Then $\mathbb{P}_p(0 \xleftrightarrow[\mathbb{R}]{} \infty) > 0$.*

Let us recall the following fact, that is a particular case of Proposition 2 (or of Lemma 2, since $(+1)$ -particles can only be hit by particles starting on their right).

Lemma 4. *Assume $p > 0$. Then $\mathbb{P}_p(0 \xleftrightarrow[\mathbb{R}_+]{} \infty \mid v_0 = +1) = 0$.*

Proof of the proposition. The proof proceeds by a definition of an exploration of the configuration on the right of 0. This exploration will produce a sequence of random locations $K_0, K_1, \dots \in \mathbb{Z}$ along with random signs ε_0, \dots , and $\tilde{\varepsilon}_0, \dots$. The locations K_n will be predictable stopping locations, in the sense that $\{K_n = k\} \in \mathcal{F}_{k-1} := \sigma((x_j, v_j); j = 0, \dots, k-1)$ for all $k \in \mathbb{N}$. And the signs ε_n will account for the number of surviving particles from x_0 to x_{K_n} with speed either 0 or -1, cf. (2). The signs $\tilde{\varepsilon}_n$ are introduced for a technical reason explained later.

Let $K_0 = 0$. Then, for $n \geq 0$, given K_n , let us define K_{n+1} , ε_n and $\tilde{\varepsilon}_n$ as follows:

- if $v_{K_n} = 0$, then $\varepsilon_n = \tilde{\varepsilon}_n = +1$ and $K_{n+1} = K_n + 1$;
- if $v_{K_n} = +1$, then $\varepsilon_n = \tilde{\varepsilon}_n = 0$, and Lemma 4 ensures that there is $k > K_n$, such that $K_n \xleftrightarrow{[K_n, k]} k$. Let k' be the least such k , and $K_{n+1} = k' + 1$;
- if $v_{K_n} = -1$, then
 - if the particle at x_{K_n} reaches 0, i.e. $K_n \xleftrightarrow[\mathbb{R}_+]{} \infty$, then $\varepsilon_n = \tilde{\varepsilon}_n = -1$ and $K_{n+1} = K_n + 1$;
 - else, there is $i \in [0, K_n)$ such that $i \xleftrightarrow{[0, K_n]} K_n$. If i was surviving before K_n , i.e. if $i \xleftrightarrow{[0, K_n]} \infty$, then $\varepsilon_n = \tilde{\varepsilon}_n = -1$ and $K_{n+1} = K_n + 1$. Else, this means there is $j \in [0, i)$ such that $j \xleftrightarrow{[0, K_n]} i$ (and we must have $v_i = 0$ and $v_j = +1$), in which case $j \xleftrightarrow{[0, K_n]} \infty$ and Lemma 4 provides $k > K_n$ such that $j \xleftrightarrow{[0, k]} k$. Let k' be the least such k , and $K_{n+1} = k' + 1$. In this last sub-case, we let $\varepsilon_n = 0$ and $\tilde{\varepsilon}_n = -1$.

This construction yields, by induction, that, for all $n \geq 1$, the sequence K_1, \dots, K_n contains all the locations of particles that survive restricted to $[0, K_{n+1})$, that none of

these surviving ones has speed $+1$, and moreover that

$$(2) \quad \sum_{0 \leq m \leq n} \varepsilon_m = \# \left(0\text{-speed particle surviving in } [0, K_{n+1}) \right) \\ - \# \left((-1)\text{-speed particle surviving in } [0, K_{n+1}) \right)$$

or, in other words,

$$\sum_{0 \leq m \leq n} \varepsilon_m = \sum_{\substack{k \in [0, K_{n+1}), \\ k \xleftrightarrow{[0, K_{n+1})} \infty}} (\mathbf{1}_{\{v_k=0\}} - \mathbf{1}_{\{v_k=-1\}}).$$

In particular, if $v_0 = 0$, then $0 \xleftrightarrow{[0, K_{n+1})} \infty$ if and only if $\varepsilon_0 + \dots + \varepsilon_m > 0$ for $m = 0, \dots, n$.

However,

$$(3) \quad \text{for all } n \in \mathbb{N}, \quad \varepsilon_n \geq \tilde{\varepsilon}_n,$$

and the sequence $(\tilde{\varepsilon}_n)_n$ is i.i.d. with law $\frac{1-p}{2}\delta_{-1} + p\delta_1 + \frac{1-p}{2}\delta_0$. Indeed, $\tilde{\varepsilon}_n$ is a function of v_{K_n} and $(v_{K_n})_n$ is i.i.d. with law μ due to the construction (recall that $(K_n)_{n \geq 0}$ are predictable stopping locations).

Assuming $p > \frac{1}{3}$, we have $\mathbb{E}_p[\tilde{\varepsilon}_0] > 0$, and a straightforward corollary of the law of large numbers implies that, with positive probability, $\tilde{\varepsilon}_0 + \dots + \tilde{\varepsilon}_n > 0$ for all n . Due to (3), we also have $\varepsilon_0 + \dots + \varepsilon_n > 0$ for all n with positive probability, hence $\mathbb{P}_p(0 \xleftrightarrow{\mathbb{R}_+} \infty \mid v_0 = 0) > 0$. Due to symmetry, and independence between positive and negative half-axes, the proposition follows. \square

Remarks.

- The argument can be modified so as to yield survival below $1/3$. One can indeed consider an exploration of locations 3 by 3, and similarly list the 27 situations and associate similar values $\varepsilon_n \in \{-3, \dots, 3\}$ to each of them, *except* to the only case $(+1, 0, -1)$, which depending on interdistances can, with equal probability, contribute to $\varepsilon_n = -1$ (if $+1$ and 0 meet first) or $\varepsilon_n = 0$ (if 0 and -1 meet first), which introduces a drift at $p = 1/3$. This argument gives survival with positive probability for $p > 0.32803\dots$
- The symmetry assumption can be dropped, if only survival on \mathbb{R}_+ is considered. Namely, assuming $\mu = q\delta_{-1} + p\delta_0 + r\delta_1$ with $p > q$, if we further assume that $(+1)$ -speed particles annihilate a.s. on \mathbb{R}_+ (as in the conclusion of Lemma 4), then the proof carries over exactly and proves that $\mathbb{P}_\mu(0 \xleftrightarrow{\mathbb{R}_+} \infty \mid v_0 = 0) > 0$; but if on the contrary a $(+1)$ -speed particle at x_0 survives with positive probability on \mathbb{R}_+ , then substituting it for a 0 -speed particle obviously preserves survival on \mathbb{R}_+ hence we still have $\mathbb{P}_\mu(0 \xleftrightarrow{\mathbb{R}_+} \infty \mid v_0 = 0) > 0$.
- Due to linear speed-change invariance (cf. Section 2), the previous remark implies that if $\mu = p\delta_\alpha + q\delta_\sigma + r\delta_\beta$ with $\alpha < \sigma < \beta$, then the assumption $p > q$ implies $\mathbb{P}_\mu(0 \xleftrightarrow{\mathbb{R}_+} \infty \mid v_0 = \sigma) > 0$.

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REFERENCES

- [1] May 2014 *Ponder This* IBM Research Challenge. <https://www.research.ibm.com/haifa/ponderthis/challenges/May2014.html>
- [2] DYGERT B., JUNGE M., KINZEL C., RAYMOND A., SLIVKEN E., AND ZHU J. (2016). The bullet problem with discrete speeds. *ArXiv prepublication ArXiv:1610.00282*
- [3] AIZENMAN, M., AND NACHTERGAELE, B. (1994). Geometric aspects of quantum spin states. *Communications in Mathematical Physics*, 164(1), 17-63.

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